S_4 CS- MAT 206

GRAPH THEORY

MODULE-I

INTRODUCTION TO GRAPHS

- Introduction
- Basic definitions
- Application of graphs
- Finite Infinite and bipartite graphs
- Incidence & Degree
- Isolated vertex
- Pendant vertex & Null graph
- Paths & Circuits
- Isomorphism
- Subgraphs
- Walks
- Paths & Circuits
- Connected graphs
- Disconnected graphs & Components

Text Book: Narsingh Deo

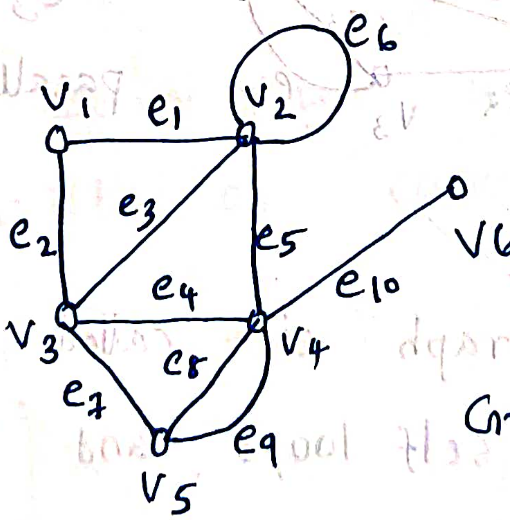
Introduction : Basic definitions

Definition of a graph:

A graph $G = (V, E)$ consists of two finite sets,

V - the vertex set of the graph, which is a non-empty set of elements called vertices &

E = the edge set of the graph, which may or may not be a non-empty set of elements called edges.



$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

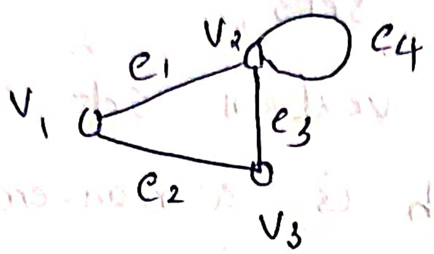
$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$

Graph with 6 vertices & 10 edges.

Vertices: are sometimes called nodes, dots.

End vertices: Each edge in E can be represented as an unordered pair of vertices called end vertices of that edge.

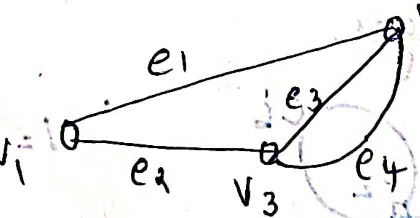
Self loop: If an edge e has identical end vertices (i.e., if the edge joins a vertex to itself) then it is called a loop.



e_4 - self loop

Parallel edges:

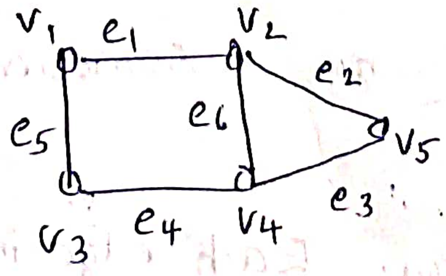
Two or more edges are said to be parallel if they have the same end vertices.



e_3 & e_4 are parallel edges.

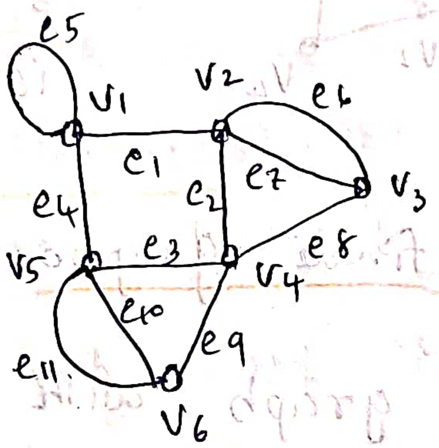
Simple Graph:

A graph is called simple if it has no self loops and no parallel edges.

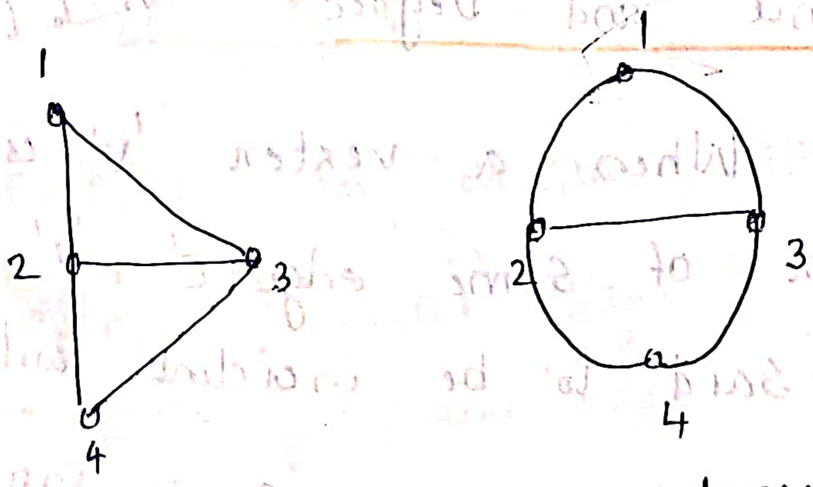


General Graph or Multigraph:

A graph with selfloops or parallel edges or both are called general graph. i.e, A graph which is not simple is called multigraph.



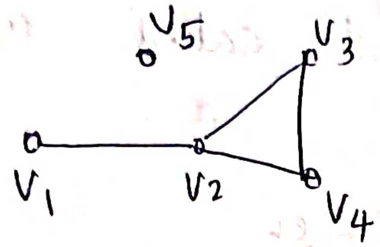
Note: In drawing a graph it is immaterial the lines are drawn straight or curved, long or short.



Same graph drawn differently.

Isolated Vertex:

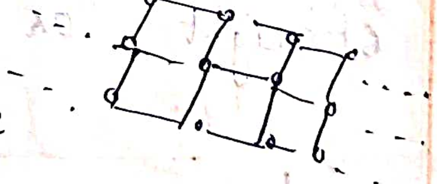
A vertex is said to be isolated if it is not an end vertex of any edge in the graph.



v_5 is an isolated vertex

Finite & Infinite graphs:

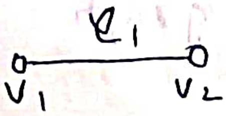
A graph with finite number of vertices as well as a finite number of edges is called a finite graph. otherwise it is an infinite graph.



infinite

Incidence and Degree

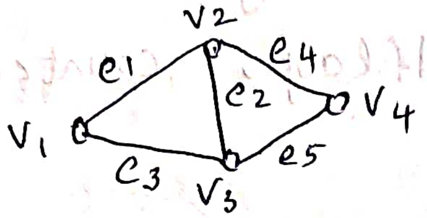
When a vertex 'v' is an end vertex of some edge 'e', 'v' and 'e' are said to be incident with each other.



e_1 is incident with v_1 & v_2

Adjacent edges:

Two non parallel edges are said to be adjacent if they are incident on a common vertex.

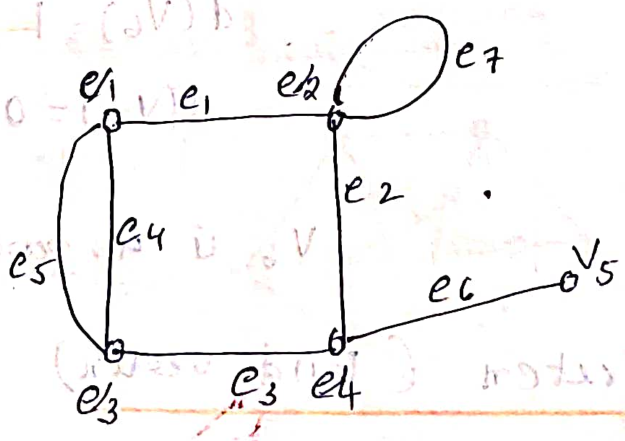


e_1 & e_3 - adjacent

$e_2\,e_5$ - adjacent

Adjacent Vertices

Two vertices which are joined by an edge are called adjacent vertices.



e_2, e_3, e_6 are incident with vertex v_4

$e_5\,e_4$ - parallel edges

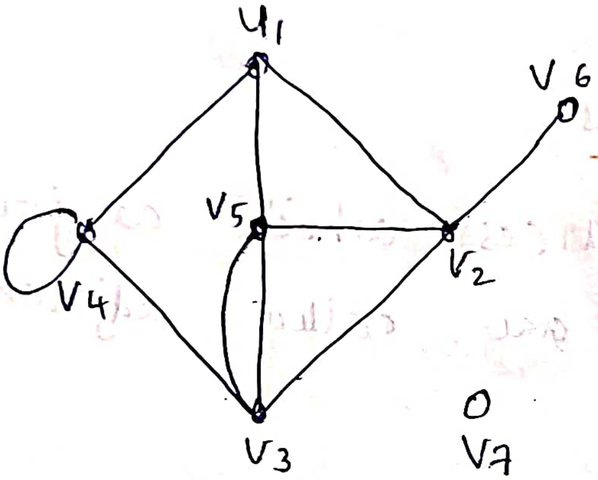
$e_1\,e_2$ - adjacent edges

$v_1\,v_2$ - adjacent vertices.

(v_1 & v_4 - non adjacent)

Degree of a vertex (Valency)

Let v be a vertex of a graph G then the degree of v denoted by d(v) is the number of edges of G incident on v with selfloops counted twice.



$d(v_1) = 3$

$d(v_2) = 4$

$d(v_3) = 4$

$d(v_4) = 4$

$d(v_5) = 4$

$d(v_6) = 1$

$d(v_7) = 0$

v_6 is a pendant vertex

Pendant Vertex (End vertex)

A vertex of degree 1 is called a pendant vertex or end vertex

Note:

For an isolated vertex v, $d(v) = 0$

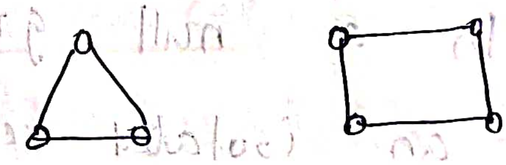
Regular Graph

A graph in which all vertices are of equal degree is called a regular graph. (or simply regular)

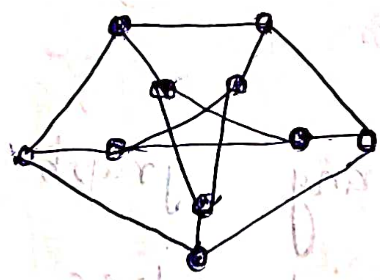
A graph is called k-regular if $d(v) = k$, $\forall\ v \in G$.

1 regular

2-regular

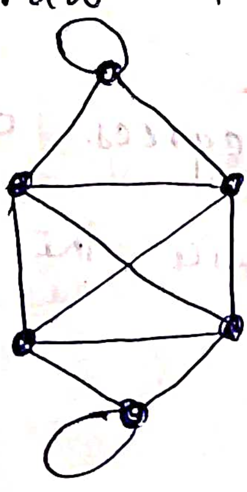


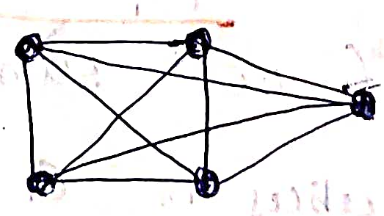
3-regular



Peterson's Graph.

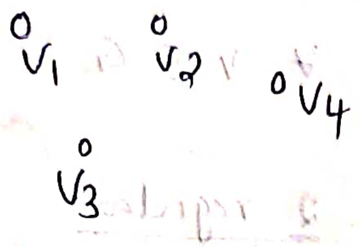
Qn Draw a regular graph of degree 4





Null graph / Empty Graph / Trivial Graph

A null graph is a graph with no edges. In a null graph the edge set is empty but vertex set is non-empty.



In a null graph every vertex is an isolated vertex.

Theorem I: First theorem of Graph Theory

Statement:

For any graph G with n vertices and 'e' edges,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

(or sum of degrees of all vertices in G is twice the number of edges in G)

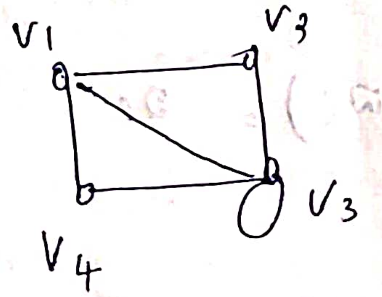
Proof

Since each edge has two end vertices it contributes two to the sum of degrees. d_i when the degrees of vertices are counted each edge is counted twice.

Hence $\sum_{i=1}^{n} d(v_i) = 2e$.

Even and odd vertices

A vertex with even degree is known as an even vertex. and a vertex with odd degree is known as an odd vertex.



$d(v_1)=3$
$d(v_2)=2$
$d(v_3)=5$
$d(v_4)=2$

v_1 & v_3 are odd degree vertices.

v_2 & v_4 are even degree vertices.

Theorem 2:

The number of vertices of odd degree in a graph is always even (ie, there is an even number of odd vertices)

Proof

Let V be the set of vertices of G

Let u_i be the set of odd vertices of G & w_i be the set of even vertices of G. & e be the number of edges in G

Then by theorem 1,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

ie, $$\underbrace{\sum d(u_i)}_{odd} + \underbrace{\sum d(w_i)}_{even} = 2e$$

$$\underbrace{\sum d(u_i)}_{odd} = 2e - \underbrace{\sum d(w_i)}_{even}$$

$$= even - even$$

$$= \text{an even number.}$$

each $d(u_i)$ is odd & R.H.S is even. Therefore there should be an even number of odd vertices

Problems:

1) Show that the maximum degree of any vertex in a simple graph with n vertices is $n-1$.

Let G be a simple graph with n vertices. Then G cannot have loops and parallel edges.

Assume the negation of R.H.S

i, Assume that G has a vertex of degree greater than or equal to n.

This is possible only if G has loops or parallel edges which is a contradiction to G is a simple graph.

Hence our assumption is wrong

a maximum degree is $< n$

$\therefore$ maximum degree of a vertex is $n-1$.

2) Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

We know that maximum degree of any vertex in a simple graph with n vertices is $n-1$

Also, $\sum d(v) = 2e \rightarrow (1)$

Consider the maximum possible values of sum of degrees,

i, $\sum d(v) = (n-1) + (n-1) + \cdots + (n-1)$

(n times, since n vertices are there)

$= n(n-1)$

sub. in eqn. (1)

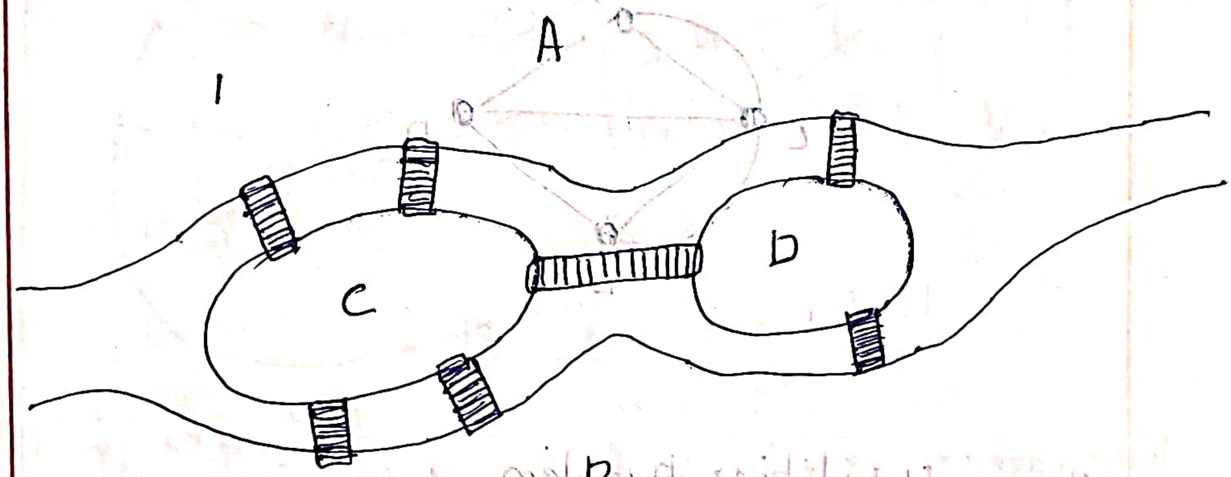
$n(n-1) = 2e$

$\therefore e = \frac{n(n-1)}{2}$

Applications of Graph

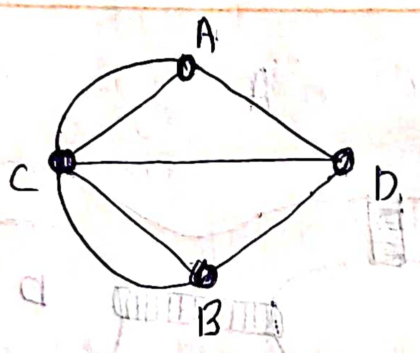
Graph Theory has a wide range of applications in Engineering, in physical social and biological sciences, in linguistics, and in numerous other area. The following are four examples from hundreds of such applications.

I. KONIGSBERG BRIDGE PROBLEM



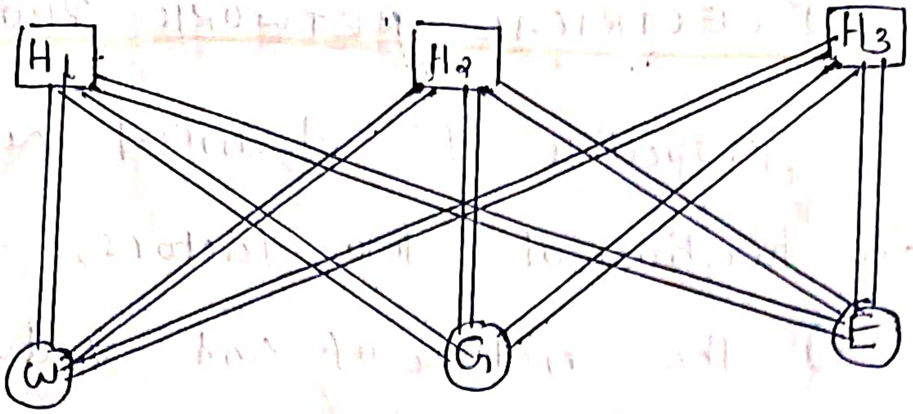
TWO Islands C & D formed by Pregel River in Konigsberg (City in PRussia) were connected to each other and to the banks of A & B with 7 bridges. The problem was to start at any of the 4 land areas of the city A, B, C, D walk over each of the 7 bridges exactly once and return to the starting point (without swimming across the river of course).

Euler represented the situation by means of a graph and proved that a solution for this problem does not exists.

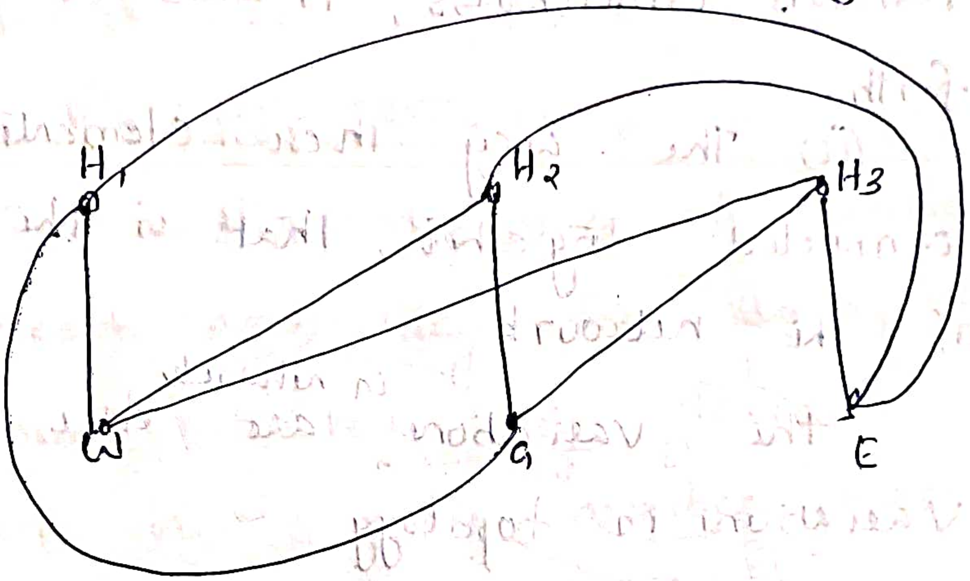


2. utilities problem

There are 3 houses H_1, H_2, H_3 each to be connected to each of the 3 utilities - water (W), Gas (G) and electricity (E) by means of conduits (tubes, wires, channels etc) Is it possible to make such connections without any crossovers of the conduits.



Graph of three utility problem.



The problem can be represented by a graph as shown above. The conduits are shown as edges while the houses and utility supply centres are vertices. It has been proved that the graph cannot be drawn in the plane without edges crossing over. Thus the answer to this problem is 'NO'.

3. ELECTRICAL NETWORK PROBLEM

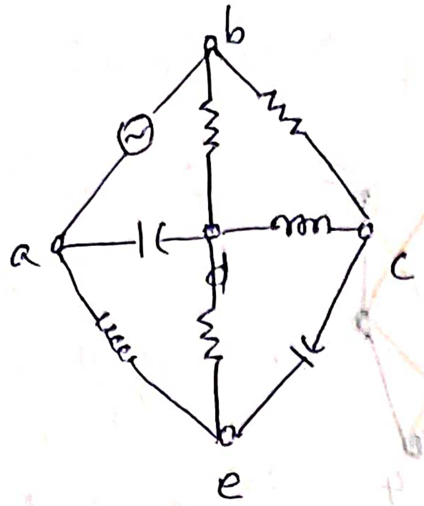
Properties of electrical networks are functions of two factors,

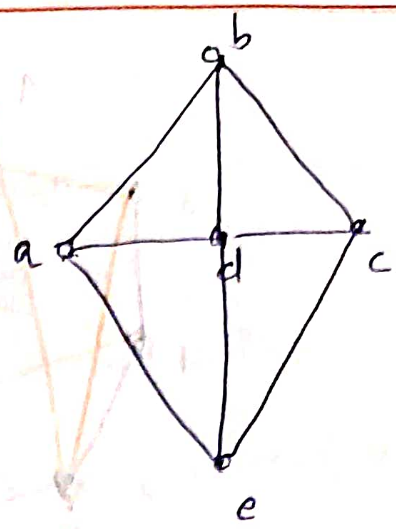
i) The nature of and value of the element forming the network such as resistors, inductors, transistors and so forth.

(ii) The way these elements are connected together, that is the topology of the network

The variations in networks are due to the variations in topology

The topology of networks is studies by means of its graphs. In drawing a graph of an electrical network the junctions are represented by vertices and branches (which consists of electrical elements) are represented by edges. An electrical network and its graph are shown below.

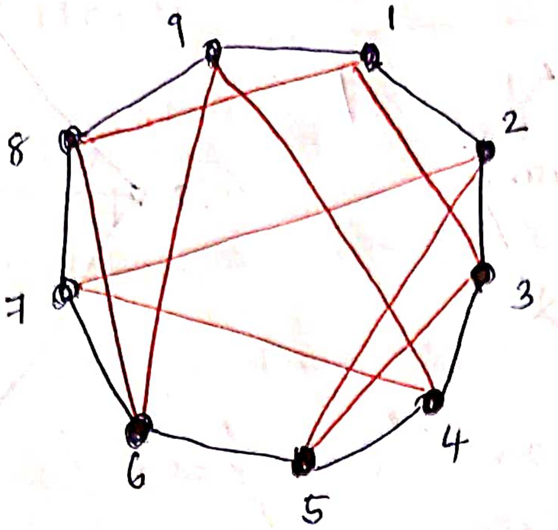




4. SEATING PROBLEM

Nine members of a new club meet each day for lunch at a round table. They decide to sit such that every member has different neighbours at each lunch. How may days this arrangement can last.?

This situation can be represented by a graph with nine vertices such that each vertex represents a member and an edge joining two vertices represents the relationship of sitting next to each other. The two possible seating arrangements shown in the figure



is, 1 2 3 4 5 6 7 8 9 1 & 1 3 5 2 7 4 9 6 8 1

It can be shown by graph theoritic ~~conditions~~ consideration, that there are only two more arrangements possible. They are 1 5 7 3 9 2 8 4 6 1 & 1 7 9 5 8 3 6 2 4 1.

In general it can be shown that for n people the number of such possible arrangements is, $\frac{n-1}{2}$, if n is odd

$\frac{n-2}{2}$, if n is even.

Complete Graph:

A complete graph is a simple graph in which each pair of distinct vertices are joined by an edge.

Complete graph with n-vertices is denoted by K_n.

K_1

K_2

1 edge

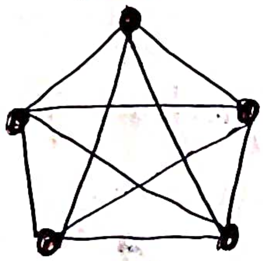
K_3

3 edge

K_4

6 edge.

K_5



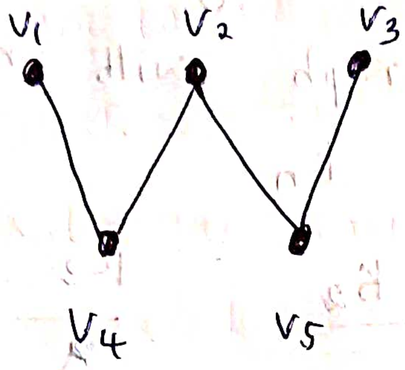
Bipartite Graph.

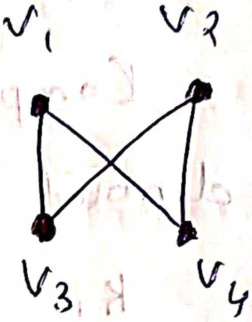
Let G be a graph. If the vertex set V of G can be partitioned into two non-empty set X & Y, i.e., $(X \cup Y = V)$ & $X \cap Y = \phi$) in such a way that each edge of G has one end in X and the other end in Y

Then G is called a bipartite graph

The partition $V = X \cup Y$ is called a partition of G.

Eg:

① 

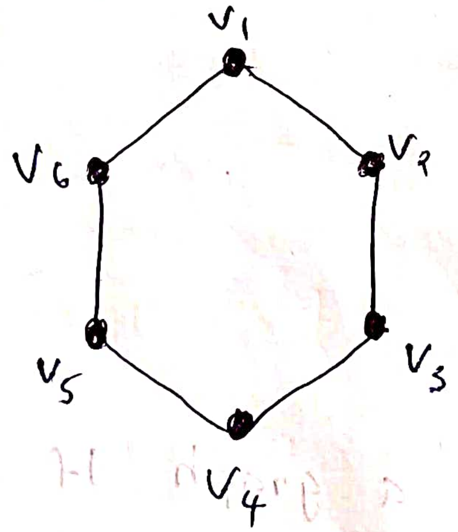
② 

$X = \{v_1, v_2, v_3\}$

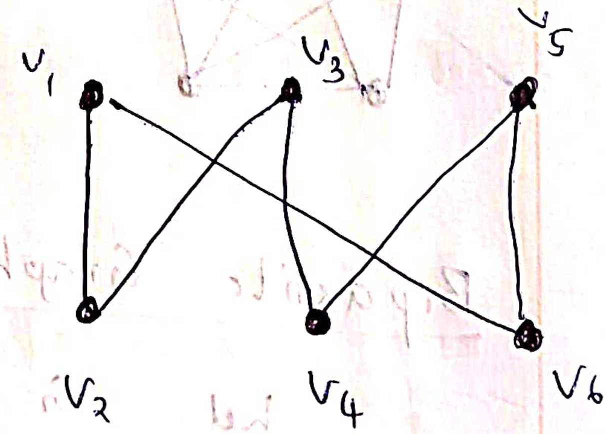
$Y = \{v_4, v_5\}$

$X = \{v_1, v_2\}$

$Y = \{v_3, v_4\}$



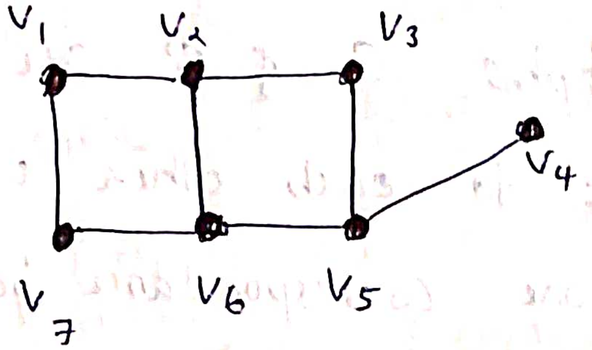
$\longrightarrow$ Bipartite graph



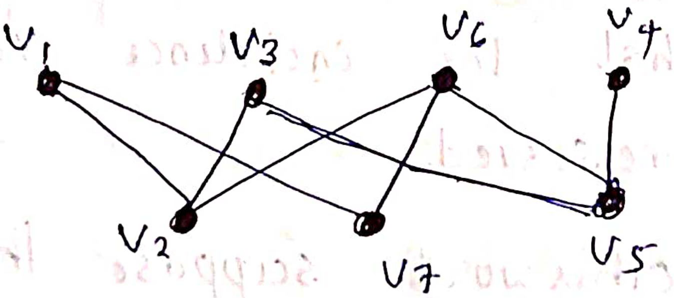
$X = \{v_1, v_3, v_5\}$

$Y = \{v_2, v_4, v_6\}$

Qn) Draw the bipartite graph of



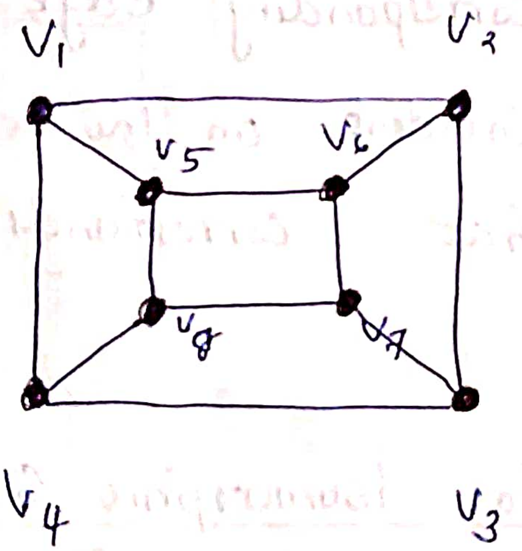
Bipartite graph



$X = \{v_1, v_3, v_4, v_6\}$

$Y = \{v_2, v_5, v_7\}$

Qn)



Note:

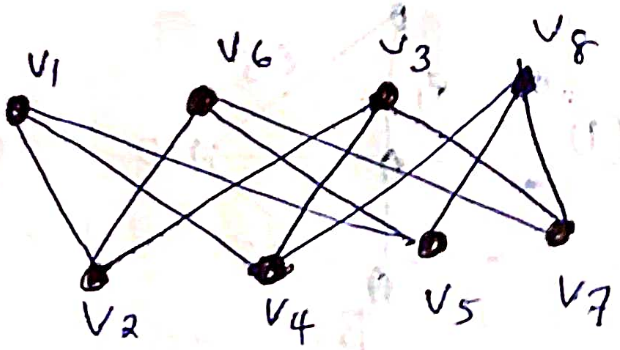
It is a 3-regular graph.

$d(v_i) = 3, \forall V$

Bipartite graph

$X = \{v_1, v_6, v_3, v_8\}$

$Y = \{v_2, v_4, v_5, v_7\}$



Complete bipartite graph

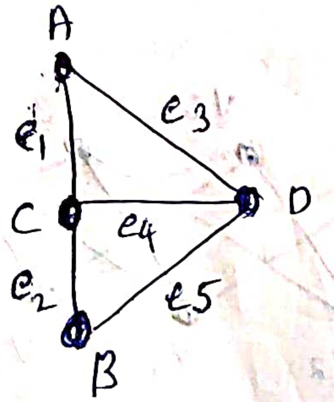
Graph Isomorphism

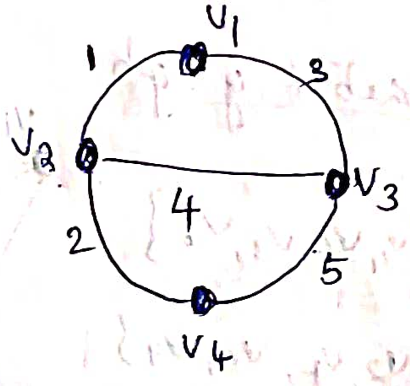
Two graphs G & G' are said to be isomorphic to each other if there is a one-to-one correspondance between their vertices and between their edges such that the incidence relationships are preserved.

In otherwords suppose that edge e is incident on vertices v_1 & v_2 on G; then the corresponding edge e' in G' must be incident on the vertices v_1' & v_2' that corresponds to v_1 & v_2 respectively.

Examples for Isomorphic Graphs

(1)



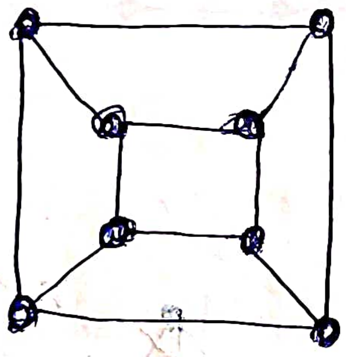


The above two graphs are isomorphic with the correspondence btw. the vertices & btw. the edges are,

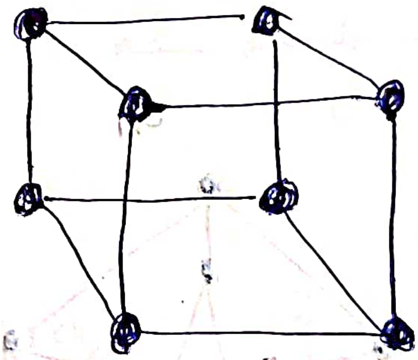
$\{A, B, C, D\}$ corresponds to $\{V_1, V_4, V_2, V_3\}$

$\{e_1, e_2, e_3, e_4, e_5\}$ corresponds to $\{1, 2, 3, 4, 5\}$ respectively.

(2)

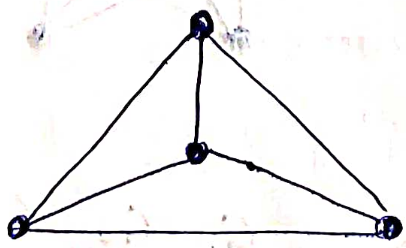


G

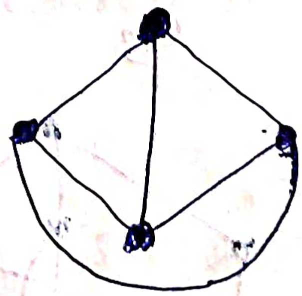


G'

(3)



G



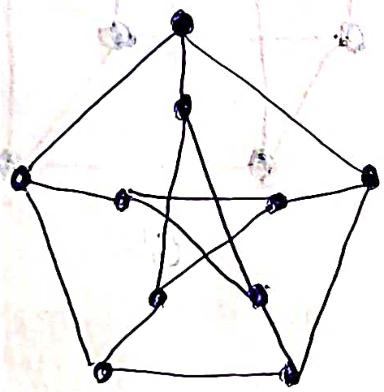
G'

Note:

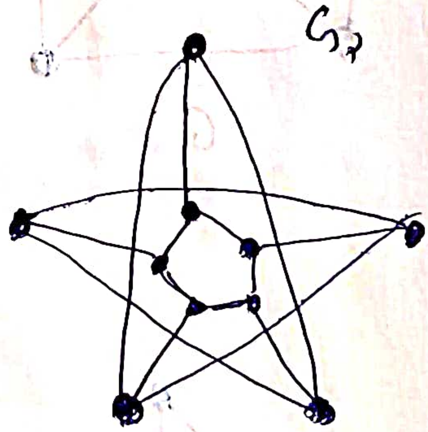
If G & G' are isomorphic they have,

1. same number of vertices
2. same number of edges
3. equal number of vertices with given degree.

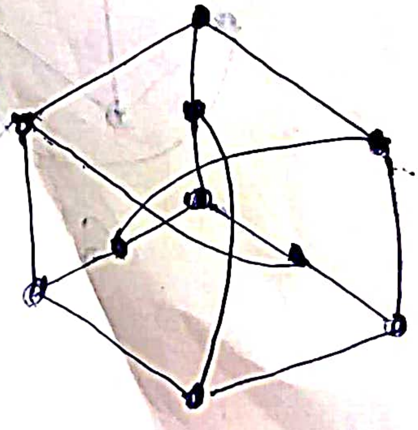
G_1



G_2



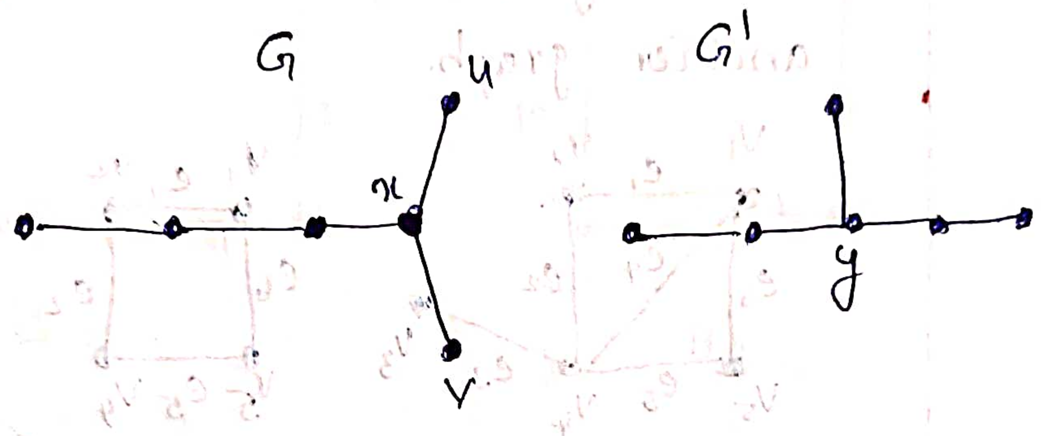
G_3



G_1, G_2, G_3 are isomorphic graphs.

But the above 3 conditions are not sufficient for isomorphic graphs. ie, there exists non isomorphic graphs with same number of edges and equal no. of vertices with a given degree & same number of vertices.

eg:



In the above graphs G & G'

no. of vertices of G & G' = 6

no. of edges of G & G' = 5

no. of vertices with degree 1 = 3

" 2 = 2

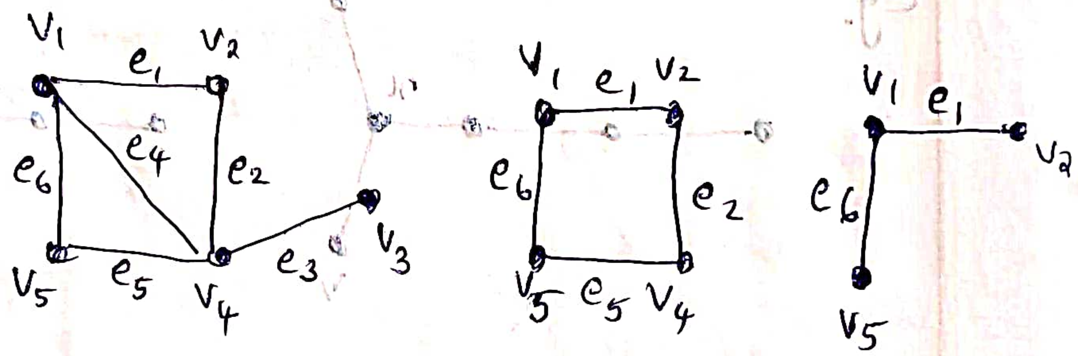
3 = 1

but they are not isomorphic. since if x corresponds to y, x has two pendant vertex adjacent to it, but y has only one pendant vertex adjacent to it.

Subgraphs

Let G and H are graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then H is a subgraph of G and G is said to be the subgraph of H.

A subgraph can be a part of another graph.



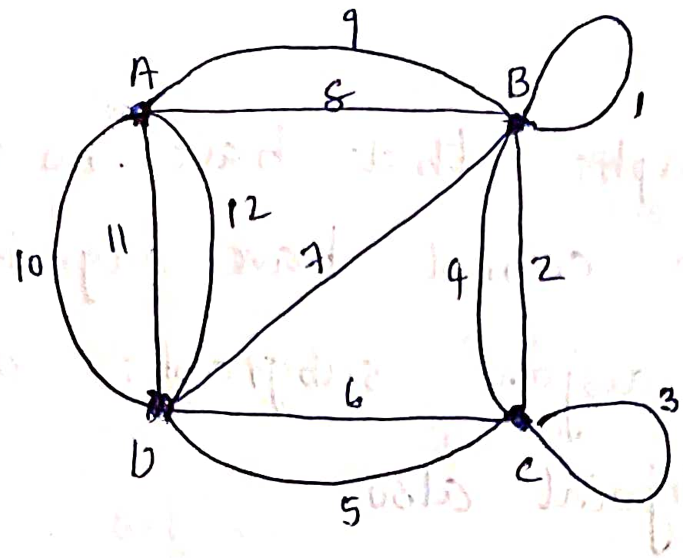
G H_1 H_2

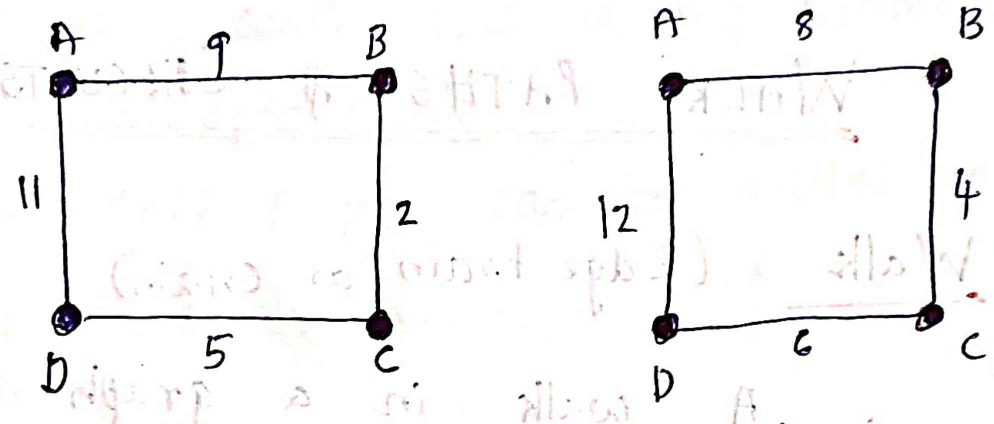
Here H_1 & H_2 are subgraphs of G.

Edge - disjoint subgraph

Two (or more) subgraphs H_1, H_2 of a graph G are said to be edge disjoint if H_1 & H_2 do not have any edge in common.

eg:

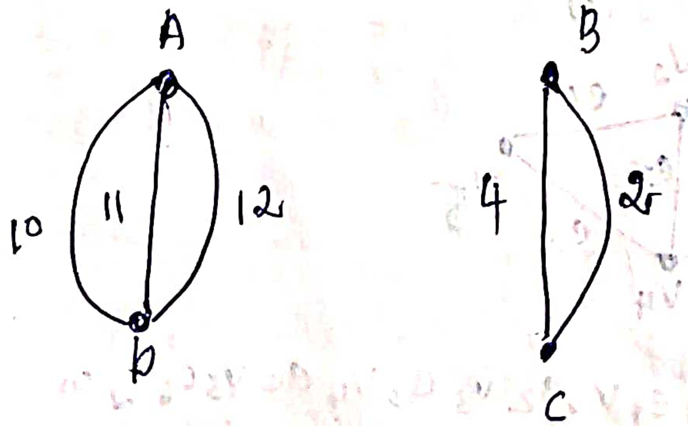




H_1 H_2

Vertex disjoint subgraphs

Subgraphs that have no vertex in common.



Note:

Subgraphs that have no vertices in common cannot have edge in common

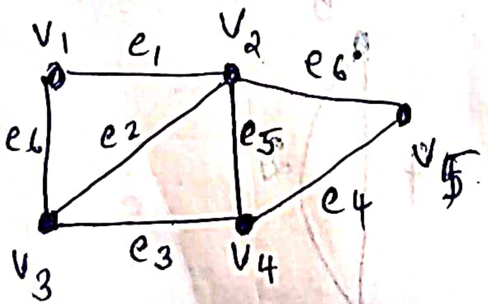
∴ Vertex disjoint subgraphs are edge disjoint also.

WALKS, PATHS & CIRCUITS

Walk : (Edge train or chain)

A walk in a graph is a finite alternating sequence of vertices and edges beginning & ending with vertices such that each edge is incident with the vertices preceding and following it.

eg:



Here $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_6 v_2$ is a walk.

- The vertices with which a walk begins & ends are called terminal vertices.
- A walk which begins & ends with same vertex is called a closed walk.
- A walk that is not closed is an open walk.
- Self loops can be added in a graph

Trail

A walk in which no edges repeats is called a trail.

Path: (simple path or elementary path)

An open walk in which no vertex appears more than once is called a path.

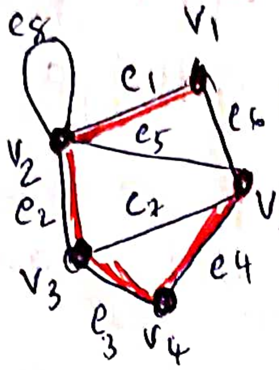
Length of the path

The number of edges in a path is called the length of the path.

Note:

Terminal vertices of a path are of degree 1 & rest of the vertices are of degree 2.

eg:



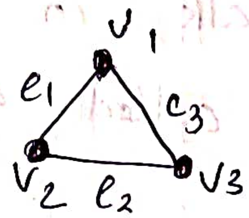
$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5$ - path

length of path - 4

($d(v_1)=1$, $d(v_5)=1$ in the path)

Circuit (Cycle / polygon / Circular path)

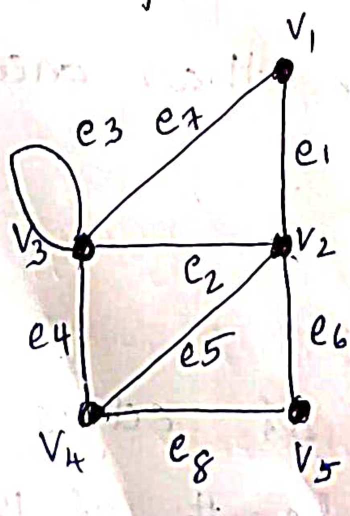
A closed path is called a circuit or cycle.



$v_1 e_1 v_2 e_2 v_3 e_3 v_1$

closed path

Examples:



$v_1 e_1 v_2 e_2 v_3 e_3 v_3 e_4 v_4 e_5 v_2 e_6 v_5$ is an open walk

$v_1 e_1 v_2 e_2 v_3 e_4 v_4$ path of length 3.

$v_2 e_2 v_3 e_4 v_4 e_5 v_2$ - circuit or cycle

Connected Graphs & Disconnected Graphs

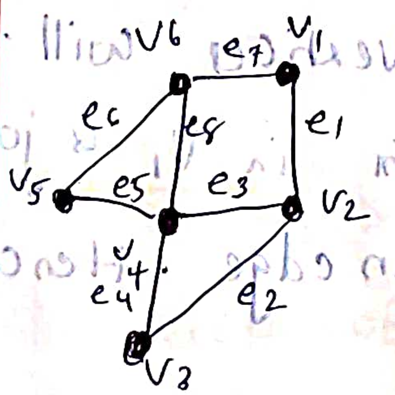
A graph G is said to be connected if there is at least one path between every pair of vertices in G. Otherwise G is disconnected.

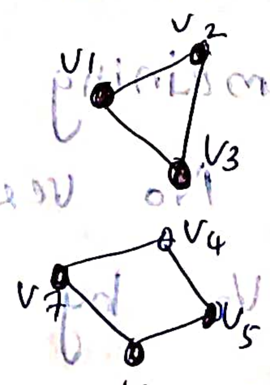
Component

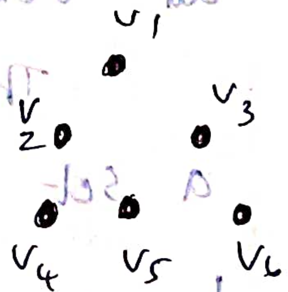
A disconnected graph consists of two or more connected subgraphs. Each of the connected subgraphs are called components.

Note:

A null graph of more than one vertex is disconnected.







Null graph

Connected

Disconnected
2 components

Disconnected
6 components

Theorem

A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 & V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in V_2.

Proof

Let G be a disconnected graph. Consider a vertex 'a' in G. Let V_1 be the set of all vertices that are joined by paths to 'a'. Since G is disconnected, V_1 does not include all vertices in G.

The remaining vertices will form a set V_2. No vertex in V_1 is joined to any in V_2 by an edge. Hence the partition.

Conversely, suppose that such a partition exists. Consider two arbitrary vertices 'a' and 'b' of G such that $a \in V_1$ & $b \in V_2$. Then no path can exists between vertices 'a' and 'b' or otherwise there would be at least one edge whose one end in V_1 & other in V_2, which is a contradiction to the existance of the partition.

Hence if a partition exists G is disconnected.

Theorem

If a graph (connected or disconnected) has exactly two vertices of odd degree there must be a path joining these two vertices.

Proof:

Let V_1 & V_2 be the two vertices of odd degree in a graph G.

If G is a connected graph

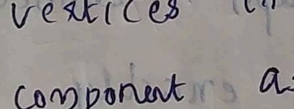
If G is a connected graph then surely there exists a path between every vertex and hence between v_1 & v_2.

Let G be a disconnected graph. If v_1 & v_2 belong to the same component of G, then they are joined by a path.

Now assume that v_1 & v_2 are in two different components. Then each component consists of a single vertex of odd degree which contradicts the theorem that there is even number of odd vertices in a graph. (considering each component as a graph)

Hence our assumption of v_1 & v_2 being in two different components is wrong.

$\therefore$ v_1 & v_2 belong to the same component and hence joined by a path.



→ one odd degree vertex in one component is not possible every vertex and hence between V_1 & V_2.

Let G be a disconnected graph. If V_1 & V_2 belong to the same component of G, then they are joined by a path.

Now assume that V_1 & V_2 are in two different components. Then each component consists of a single vertex of odd degree which contradicts the theorem that there is even number of odd vertices in a graph. (considering each component as a graph)

Hence our assumption of V_1 & V_2 being in two different components is wrong.

$\therefore$ V_1 & V_2 belong to the same component and hence joined by a path.

→ one odd degree vertex in one component is not possible

Theorem

A simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof

Let G be a graph with k components.

Let $H_1, H_2, H_3, \ldots H_k$ be the k components of G with ~~vertices~~ no. of vertices $n_1, n_2, \ldots n_k$ respectively with $n_i \geq 1$ for $i = 1, 2, 3 \ldots k$.

We know that,

$$H_1 \cup H_2 \cup \ldots \cup H_k = G$$

$$\text{i.e. } n_1 + n_2 + \cdots + n_k = n$$

$$\sum_{i=1}^{k} n_i = n \quad (1)$$

Now,

$$\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$$

$$= n - k$$

$$\text{i.e. } \sum_{i=1}^{k} (n_i - 1) = n - k$$

squaring both sides.

$$\left[\sum_{i=1}^{k}(n_i-1)\right]^2 = (n-k)^2$$

i, $\sum (n_i-1)^2$ + some of +ve terms $= (n-k)^2$

i, $$\sum_{i=1}^{k}(n_i-1)^2 \le (n-k)^2$$

$$\sum_{i=1}^{k}(n_i^2-2n_i+1) \le (n-k)^2$$

$$\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \le (n-k)^2$$

$$\sum_{i=1}^{k} n_i^2 - 2n + k \le (n-k)^2$$

i, $$\sum_{i=1}^{k} n_i^2 \le (n-k)^2 + (2n-k) \longrightarrow (1)$$

Now the maximum number of edges of the i^{th} component H_i with n_i vertices is, $\frac{n_i(n_i-1)}{2}$

$\therefore$ Maximum no. of edges in G

$$= \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \dots + \frac{n_k(n_k-1)}{2}$$

$$= \frac{1}{2} \sum_{i=1}^{k} (n_i(n_i-1))$$

$$= \frac{1}{2} \left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i \right]$$

$$= \frac{1}{2} \left[\sum_{i=1}^{k} n_i^2 - n \right]$$

$$\leq \frac{1}{2} \left[(n-k)^2 + (2n-k) - n \right] \quad \text{from } (1)$$

$$\leq \frac{1}{2} \left[(n-k)^2 + (n-k) \right]$$

$$\leq \frac{1}{2} (n-k)(n-k+1)$$

Problems

1. Show that there is no simple graph with 7 vertices, one of which has degree 2, two have degree 3, three have degree 4 and remaining vertex have degree 5.

We know that $\sum d(v_i) = 2e$

i.e., $2 + (3+3) + (4+4+4) + 5 = 25$

i.e., $25 = 2e$

i.e., $e = 25/2 = 12.5$ which is not possible.

$\therefore$ No such graph exists.

2. Construct a simple graph of 12 vertices with two of them have degree 1, three having degree 3 and remaining 7 vertices having degree 10.

3 vertices having odd degree. Odd no. of odd degree vertices not possible.

Hence such a graph does not exist.

(3) Let G be a graph on n vertices & m edges. Suppose t of the vertices are of degree 'k' and remaining n-t vertices are of degree k+1. Prove that, $t = (k+1)n - 2m$

By Ist theorem of graph theory

$$\sum d(v) = 2e, \quad \text{here } e = m$$

i, $t \times k + (n-t)(k+1) = 2m$

i, $tk + nk + n - tk - t = 2m$

$$n(k+1) - t = 2m$$

$$\therefore t = n(k+1) - 2m$$

(4) Prove that it is impossible to have a group of 9 people at a party such that each one knows exactly 5 of the others in the group.

Consider this as a graph where people are vertices and the relation as ~~vertex~~ as knowing the people as edges.

Then each of the 9 vertices has exactly 5 edges incident on it.

i.e., $\sum d(v) = 9 \times 5$

$= 45$

it is not possible because, $2e = 45$ is not possible.

Hence the result.

---

5. What is the largest number of vertices in a graph with 35 edges of all vertices are of degree at least 3.

Here $e = 35$.

We know, $\sum d(v) = 2e = 35$

Let n be the no. of vertices in such a graph, then each vertex has at least 3 degree

i.e., $3n \le \sum d(v_i) = 35 \times 2 = 70$

i.e., ~~$3n \le 35 \times 2$~~ i.e., $n \le 70/3 = 23.3$

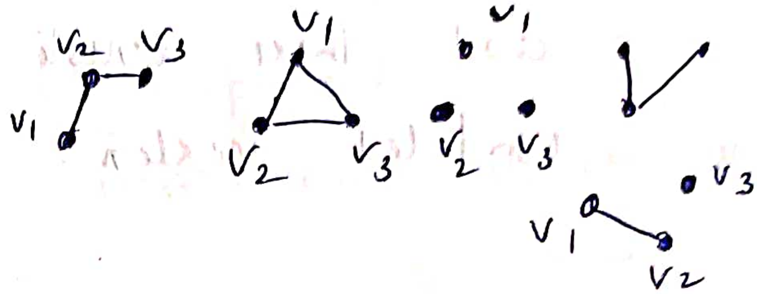
So the graph has at most 23 vertices

6. Draw all simple graphs of one, two three and four vertices.

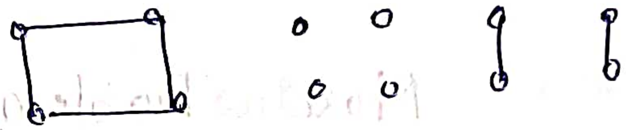
One vertex :

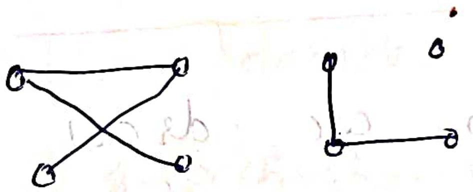
2 vertices :

3 vertices :



4 vertices :





etc.

7. Let G be a graph with n vertices and n-1 edges. P.T. G has either a pendant vertex or an isolated vertex.

Let us assume that G has neither a pendant vertex nor an isolated vertex.

which implies, $d(v) \geq 2$, for all vertices

i, $\sum d(v) \geq 2n$

i, $2e \geq 2n$

i.e. $e \geq n$

which is a contradiction since

G has $n-1$ edges only.

Hence our assumption was wrong

and there exists a pendant or

isolated vertex.

More Problems

1) Can we draw a graph with 3 vertices v_1, v_2 & v_3 where;

a) $d(v_1) = 1$, $d(v_2) = 2$, $d(v_3) = 2$

b) $d(v_1) = 2$ $d(v_2) = 1$ $d(v_3) = 1$

2) Determine the number of edges in a graph with 6 vertices, 2 of degree 4 and 4 of degree 2. Draw such 2 graphs.

3) Draw a connected graph that becomes disconnected when any edge is removed from it.

Imp.
uQ 4. Draw a disconnected graph G with 10 vertices & 4 components and also calculate the maximum number of edges possible.

5. Consider a graph with 4 vertices and degrees of vertices v_1, v_2, v_3, v_4 are 3, 5, 2, 1 respectively. Is it possible to construct such a graph. If not why?

uQ 6. Write any two application of graphs with sufficient explanation.

7. Define pendant vertex, isolated vertex and null graphs with examples of each.

8. Prove that number of vertices of odd degree in a graph is always even

9. S.T. in a simple graph with n vertices the maximum number of edges is $\frac{n(n-1)}{2}$ and maximum degree of any vertex is $n-1$

10. Define subgraphs. Give examples

11. There are 37 telephones in a city Is it possible to connect them with wires so that each telephone is connected with exactly 7 others? Substantiate your answer with graph concept.

12. What are the basic conceptual conditions to be satisfied for two graphs to be isomorphic

13. Determine a walk, trail, path & cycle in the graph below.